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Radiation from a class of string theoretic black holes

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The emission of a scalar with low energy ω , from a D ($4 \leq D \leq 8$) dimensional black hole with n charges is studied in both string and semiclassical calculations. In the lowest order in ω , the weak coupling string and semiclassical calculations agree provided that the Bekenstein–Hawking formula is valid and the effective central charge $c_{\text{eff}} = 6$ for any D . When the next order in ω is considered however, there is no agreement between the two schemes unless $D = 5$, $n = 3$ or $D = 4$, $n = 4$.

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1 Introduction

It has been over twenty years since the Bekenstein–Hawking formula of black hole entropy was presented, but until very recently there has been no precise statistical mechanics explanation of this. The possibility of a microscopic explanation of black hole entropy in terms of the counting of string states was first suggested a few years ago in [1] and was further developed in [2]. In the past year we have seen, for certain black holes within the context of type II superstring theory, that the number of string states correctly gives the black hole entropy geometrically calculated using the Bekenstein–Hawking formula. First, this agreement was shown for a five dimensional ($D = 5$) black hole carrying three types of charge ($n = 3$) in the extremal configuration [3]. Then it was extended to non-extremal configurations [4]–[6]. The agreement was also confirmed for $D = 4$, $n = 4$ black hole in both extremal and non-extremal configurations [7]–[9]. The analysis in string theory is based on D-brane technology at weak coupling whereas black hole configurations are studied in the strong coupling regime. In the case of extremal black holes, which correspond to BPS saturated bound states of D-branes, there is a topological invariance which preserves the degeneracy of states as the coupling is varied. This is not the case, however, for non-BPS string configurations and non-extremal black holes.¹

String theory knows not only about thermodynamic quantities such as entropy, but also about dynamical processes such as emission from a black hole. Emission processes inevitably include a non-extremal configuration in which a black hole emits, for example, scalar particles and falls into an energetically stable extremal configuration. Das and Mathur [11] calculated the emission rate of a scalar of energy ω from a system of D-branes which corresponds to the $D = 5$, $n = 3$ black hole and showed that the string calculation and the semiclassical black hole calculation agree under the condition, $T_R \ll T_L$. (T_R , T_L are the temperatures of right moving modes and left moving modes, respectively.) Maldacena and Strominger [12] dropped the condition and showed in a semiclassical

¹Recently, Maldacena showed that at sufficiently low energies the weak coupling calculation agrees with strong coupling black hole calculation for a class of near extremal five-dimensional black holes [10].

calculation that the absorption cross section factorizes as,

$$\sigma_{\text{abs}} \propto \frac{\rho(\omega/2T_L)\rho(\omega/2T_R)}{\rho(\omega/T_H)}, \quad (1.1)$$

where $\rho(x) = (e^x - 1)^{-1}$ is the Bose distribution function, and T_H is the Hawking temperature. They further showed that the corresponding emission rate,

$$\Gamma = \sigma_{\text{abs}} \rho\left(\frac{\omega}{T_H}\right) \frac{d^4 k}{(2\pi)^4}, \quad (1.2)$$

agrees with the string calculation. In this agreement between the semiclassical calculation and the string calculation, the factorizability of the absorption cross section in the semiclassical calculation is essential. (The factor $\rho(\omega/2T_L)\rho(\omega/2T_R)$, which is the product of the distribution of left moving modes and that of the right moving ones, naturally appears in the string calculation.)

In order to extend the discussion to other values of D and n , let us introduce the parameter

$$\lambda = \frac{D-2}{D-3} - \frac{n}{2}. \quad (1.3)$$

λ becomes zero only when $D = 5$, $n = 3$ and $D = 4$, $n = 4$ if $4 \leq D \leq 8$. These two combinations of spacetime dimension and the number of charges are special because the corresponding black holes have a regular event horizon and finite horizon area in the extremal limit [13][14].

In this paper we try to generalize the results explained above to the $\lambda \neq 0$ case to see whether an effective string model can describe the properties of black holes in arbitrary ($4 \leq D \leq 8$) dimensions with different numbers of charges. A priori one might expect this to work because the Bekenstein–Hawking entropy is $O(\hbar^{-1})$ and that is precisely the behavior of the entropy of a one dimensional gas of massless particles. As in [15] the emission rate at lowest order in the energy is compared with the semiclassical calculation of Hawking radiation by using the result of [16]. Now while Das and Mathur’s work [11] is based on D-brane technology, however, one can formulate the entire calculation of emission rate and absorption cross section for a D -dimensional black hole by considering an effective string model. For the purposes of this paper we do not need to specify how

this effective string is built up of configurations of D-branes, NS-NS-branes, M-branes etc. We find that the Bekenstein–Hawking formula is recovered provided that the effective central charge c_{eff} is six.² Since $c_{\text{eff}} = 6$ corresponds to the physical degrees of freedom in six dimensional string theory whilst the above is true for general D , $4 \leq D \leq 8$, this result is somewhat puzzling. In the $D = 5$, $n = 3$ black hole which is described in terms of D-1-branes bound to D-5-branes there is some understanding of this effective value of the central charge [18]. A similar argument has also been given in the case of $D = 4$, $n = 4$ black hole [18] though there it appears to be more tentative.³ However in the general case it is unclear why one should have this value for the effective central charge.

Since the string calculation of emission rate can be generalized to D -dimensions, it is natural to ask whether one can also generalize the calculation of [12] to D -dimensions, i.e., whether the absorption cross section factorizes in arbitrary D -dimensions as it does in five dimensions.⁴ We show that the factorization does not occur unless $\lambda = 0$. This implies that the string calculation and the semiclassical calculation do not agree when $\lambda \neq 0$ beyond the lowest order in ω . Perhaps this is an indication that the model makes sense only in the $\lambda = 0$ case where indeed there is some explanation of why $c_{\text{eff}} = 6$. However we hesitate to draw such a strong conclusion since there is no reason why one should believe that the weak coupling string calculation should agree beyond extremality even in the $\lambda = 0$ case.⁵ It was a surprise even there and the agreement of the lowest order in ω calculation for $\lambda \neq 0$ is also a surprise. What is not a surprise is the fact that it does not agree beyond the lowest order.

In the next section, we introduce our model. We also derive some thermodynamic relations for later use. In section 3, we calculate the emission rate and the absorption cross section from a string configuration and from a black hole in D -dimensions, and see that the two results agree in the lowest order in energy if $c_{\text{eff}} = 6$ irrespective of D .

²The same conclusion has been reached by Halyo et al [15][17] for all black holes.

³For a different perspective, see [19] and [20].

⁴The absorption cross section factorizes also in $D = 4$, $n = 4$ case [21].

⁵See footnote 1.

In section 4, higher order contributions are considered and it is shown that the string calculation and the semiclassical calculation agree only when $\lambda = 0$. We summarize our conclusions in the last section.

2 The Effective String Model

We consider a string configuration in type II (either A or B) theory, described by the sum of the world sheet σ -model action,

$$I = \frac{1}{2\pi\alpha'} \int d^2\sigma \frac{1}{2} \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}, \quad (2.1)$$

and the low energy effective action,

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\phi} [R + \dots]. \quad (2.2)$$

The σ -model action I describes an effective string source for the background field configuration. We expand the metric around a black string background $g_{\mu\nu}^0 = \eta_{\mu\nu} + O(\kappa^2)$ as

$$g_{\mu\nu} = g_{\mu\nu}^0 + \sqrt{2}\kappa h_{\mu\nu}. \quad (2.3)$$

κ is the ten-dimensional gravitational coupling constant, which is proportional to the string coupling constant g . The second term describes the propagation of the graviton. At strong coupling, $g_{\mu\nu}^0$ should be the black string background in which gravitons propagate. At weak coupling, on the other hand, one can neglect $O(\kappa^2)$ term and background spacetime becomes flat with a small fluctuation represented by the second term on the r.h.s. of eq.(2.3), which describes the emission from the string.

Let us look at the mass formula for a string with one dimension compactified on a circle of radius R (or the circumference $L = 2\pi R$).

$$E^2 = Q_L^2 + 8\pi T N_L = Q_R^2 + 8\pi T N_R, \quad (2.4)$$

where

$$Q_L^2 = \left(n_w L T - \frac{2\pi n_p}{L} \right)^2, \quad Q_R^2 = \left(n_w L T + \frac{2\pi n_p}{L} \right)^2. \quad (2.5)$$

$T = 1/2\pi\alpha'$ is the fundamental string tension, n_w is the winding number, n_p is the Kaluza-Klein mode. The matching condition $N_L - N_R = n_w n_p$ must be satisfied. Following [22] and [23], we make the long string approximation,

$$\frac{N_{L,R}}{L^2 T} \ll 1. \quad (2.6)$$

The meaning of this will become clear soon. We split the energy E into that for the left movers and that for the right movers under the long string approximation, eq.(2.6),

$$E_L = \frac{1}{2}(E - E_0 + P) = \frac{2\pi}{n_w L} N_L + O(L^{-2}), \quad (2.7)$$

$$E_R = \frac{1}{2}(E - E_0 - P) = \frac{2\pi}{n_w L} N_R + O(L^{-2}), \quad (2.8)$$

where $E_0 = n_w L T \gg E_{L,R}$ is the ground state energy, $P = 2\pi n_p / L = E_L - E_R$ is the total momentum flowing along the string. E_L and E_R describe the energy associated with the left and right moving oscillator excitations of the string. After rescaling $n_w L \rightarrow L$, we have

$$E = E_0 + E_L + E_R = LT + \frac{2\pi}{L}(N_L + N_R) + O(L^{-2}). \quad (2.9)$$

Since the black hole at strong coupling is expressed in terms of the string sitting at the origin, its mass should be given by $M = E \simeq LT$ in the lowest order. The fact $E_0 \gg E_{L,R}$ under the long string approximation means that the oscillator excitations are small compared with the ground state energy. Thus we can expect that the excitations are sparsely distributed on the long string and that there is little interaction between them.

In the weak coupling limit, the oscillator excitations can be considered as a one dimensional gas moving on the long string with independent left and right moving modes. We summarize its thermodynamics in the rest of this section. The emission of the graviton following the interaction of left and right movers is an effect of order κ . (See eq.(2.3).) In the lowest order, the right moving and the left moving sectors don't interact each other, but each sector is in thermal equilibrium at a temperature either T_L or T_R . Thus any thermodynamic quantity O can be split into two parts, O_L and O_R , for the left and right

moving sectors, respectively. The free energy of a gas of one bosonic species is given by $-\pi LT_{L,R}^2/12$. We are studying a supersymmetric system with f bosonic species and f Majorana-fermionic ones. The effective central charge is $c_{\text{eff}} = 3f/2$. Thus the total free energy is given by

$$F_{L,R} = -\frac{\pi}{12}LT_{L,R}^2 \cdot \frac{3}{2}f. \quad (2.10)$$

The entropy is

$$S_{L,R} = -\frac{\partial F_{L,R}}{\partial T_{L,R}} = \frac{\pi}{6}LT_{L,R} \cdot \frac{3}{2}f. \quad (2.11)$$

The energy of the system is

$$E_{L,R} = F_{L,R} + T_{L,R}S_{L,R} = \frac{\pi}{12}LT_{L,R}^2 \cdot \frac{3}{2}f. \quad (2.12)$$

From the expressions above, we have the relations between $T_{L,R}$, $E_{L,R}$ and $S_{L,R}$;

$$T_{L,R} = \sqrt{\frac{8E_{L,R}}{\pi f L}} = \frac{4S_{L,R}}{\pi f L}. \quad (2.13)$$

When the total momentum is fixed, i.e., $\delta P = \delta E_L - \delta E_R = 0$, we see $\delta E|_P = \delta E_L + \delta E_R = 2\delta E_L = 2\delta E_R$. Therefore we have,

$$\frac{1}{T_H} = \frac{\partial S}{\partial E} = \frac{\partial S_L}{2\partial E_L} + \frac{\partial S_R}{2\partial E_R} = \frac{1}{2} \left(\frac{1}{T_L} + \frac{1}{T_R} \right). \quad (2.14)$$

There is one more useful relation, which is obtained from eqs.(2.13) and (2.14),

$$T_L T_R = \frac{2}{\pi f L} T_H S. \quad (2.15)$$

Finally, from eqs.(2.7), (2.8) and (2.13), we recover the well known expression,

$$S = S_L + S_R = 2\pi \sqrt{\frac{c_{\text{eff}}}{6}} \left(\sqrt{N_L} + \sqrt{N_R} \right). \quad (2.16)$$

3 Lowest Order Analysis

3.1 String Calculation

Following [11], we consider an emission process in which a right mover with momentum p and a left mover with momentum q collide and emit a scalar into the $(D+1)$ -dimensional space, coming from graviton polarized in the compact directions, with momentum k . The S -matrix element for this process is

$$S_{fi} = (2\pi)^2 \delta^2(p + q - k) \frac{-iA}{\sqrt{(2p_0 L)(2q_0 L)(2k_0 V_9)}}. \quad (3.1)$$

V_9 is the nine-dimensional spatial volume. A is the amplitude determined by $I + S$. We rescale X^μ to $X^\mu = \sqrt{2\pi\alpha'} \bar{X}^\mu$ in order to get the standard normalization of the kinetic term. Also we set the world sheet to be flat, i.e., $\gamma^{\alpha\beta} = \delta^{\alpha\beta}$,

$$I = \int \frac{1}{2} \partial_\alpha \bar{X}^\mu \partial_\alpha \bar{X}^\nu \eta_{\mu\nu} + \kappa \int \partial_\alpha \bar{X}^\mu \partial_\alpha \bar{X}^\nu h_{\mu\nu}. \quad (3.2)$$

We consider the emission of graviton which is polarized, for example, in 6 and 7 directions (when $D = 4$). We also rescale h_{67} to $h_{67} = \bar{h}_{67}/\sqrt{2}$, which gives the correct normalization of the kinetic term for \bar{h}_{67} in the low energy effective action, S , eq.(2.2). This modifies the second term in eq.(3.2),

$$\sqrt{2}\kappa \int \partial_\alpha \bar{X}^6 \partial_\alpha \bar{X}^7 \bar{h}_{67}. \quad (3.3)$$

This fixes the strength of the interaction and gives, to lowest order, the amplitude, $A = \sqrt{2}\kappa p \cdot q$, as in [11]. Thus we recover the S -matrix element given in [11] when comparing to IIB compactified to five dimensions. Note that we have shown the duality between fundamental string and D-string by explicitly showing that the two formulations give the same S -matrix element. We consider a D -dimensional spacetime obtained by the compactification on $T^{9-D} \times S_1$ with volume $V_{9-D}L$. The string of length L is wrapped around S_1 . One can calculate the emission rate of the scalar of energy ω . The calculation

is given in [11] for $D = 5$ and the generalization to D -dimensions is straightforward. One gets,

$$\Gamma_{\text{str}} = \frac{L\kappa_D^2}{4}\omega\rho\left(\frac{\omega}{2T_L}\right)\rho\left(\frac{\omega}{2T_R}\right)\frac{d^{D-1}k}{(2\pi)^{D-1}}. \quad (3.4)$$

Here, $\kappa_D^2 = \kappa^2/LV_{9-D} = 8\pi G_D$, G_D being the D -dimensional Newton's constant. The luminosity is given by

$$P_{\text{str}} = \frac{L\kappa_D^2}{4}\omega^2\rho\left(\frac{\omega}{2T_L}\right)\rho\left(\frac{\omega}{2T_R}\right)\frac{d^{D-1}k}{(2\pi)^{D-1}} \quad (3.5)$$

$$\simeq \frac{16G_DS}{f}T_H|K|^{-2}\frac{d\omega}{2\pi} \quad (\omega \rightarrow 0). \quad (3.6)$$

In the last step, we expanded the Bose distribution functions in small ω and used eq.(2.15). We also used the notation, $|K|^2 = (2\pi)^{D-2}\omega^{-(D-2)}\Omega_{D-2}^{-1}$, where $\Omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)$ is the $(D-2)$ -dimensional area of the unit $(D-2)$ -sphere.

Next, we consider an absorption process. The five dimensional case was worked out in [24]. One can remove the restriction $T_R \ll T_L$ used in [24] and generalize the calculation to the D -dimensional case. Since this is exactly the inverse process of the emission process considered above, we use the same notations. A scalar in $(D+1)$ -dimensions with energy ω and no momentum in the string direction, polarized in $(9-D)$ -dimensional compact directions, is absorbed by a string, creating one left moving excitation with momentum p and one right moving excitation with momentum q . From the momentum conservation on the string world sheet, $p_0 = q_0 = \omega/2$. The absorption cross section is defined by

$$\sigma_{\text{abs}} = 2\mathcal{R}\mathcal{F}^{-1}, \quad (3.7)$$

where \mathcal{R} is the absorption rate, $\mathcal{F} = \rho(\omega/T_H)V_{D-1}^{-1}$ is the flux of incident wave. V_{D-1} is the volume of non-compact $(D-1)$ -dimensional spatial directions transverse to the string. The factor 2 comes from the interchange of polarizations assigned on the two string excitations. Following [24], \mathcal{R} is given by,

$$\mathcal{R} = \frac{2\pi|R|^2}{\Delta E}\rho\left(\frac{\omega}{2T_L}\right)\rho\left(\frac{\omega}{2T_R}\right). \quad (3.8)$$

We included the distribution functions of the right and left moving excitations. R is the amplitude to excite the string to any one of the excited levels separated by $\Delta E =$

$(2\pi/L) \times 2$ (see eq.(2.9)),

$$R = \sqrt{2}\kappa|p_1| \frac{1}{\sqrt{2\omega L V_{9-D} V_{D-1}}}. \quad (3.9)$$

We split the nine-dimensional spatial volume V_9 into the product of the length of the string, L , the volume of $(9-D)$ -dimensional compact space, V_{9-D} , and the volume of $(D-1)$ -dimensional tangential space, V_{D-1} . Using $\kappa^2 = L V_{9-D} \kappa_D^2$, one obtains

$$\sigma_{\text{abs}} = \frac{L\kappa_D^2}{4} \omega \frac{\rho(\omega/2T_L)\rho(\omega/2T_R)}{\rho(\omega/T_H)}. \quad (3.10)$$

3.2 Semiclassical Calculation

Das et al [16] studied a general D -dimensional metric of the form

$$ds_D^2 = -f(r)dt^2 + g(r)[dr^2 + r^2 d\Omega_{D-2}^2]. \quad (3.11)$$

This metric can describe D -dimensional black holes with arbitrary number of charges. They studied the propagation of a scalar field in this black hole background. The absorption probability of s-wave is given by eq.(13) in [16],

$$|A|^2 = \frac{A_H}{|K|^2}, \quad (3.12)$$

from which one can calculate the luminosity,

$$P_{\text{sc}} = \frac{d\omega}{2\pi} \frac{\omega |A|^2}{e^{\omega/T_H} - 1} \simeq \frac{d\omega}{2\pi} T_H |A|^2 = A_H T_H |K|^{-2} \frac{d\omega}{2\pi} \quad (\omega \rightarrow 0). \quad (3.13)$$

By comparing it with the string calculation eq.(3.6), we obtain $A_H = 16G_D S/f$. This implies $f = 4$ (i.e. $c_{\text{eff}} = 6$) irrespective of D in order for the Bekenstein–Hawking formula to hold. This conclusion is very general. No matter how many charges a black hole carries, and independent of D ($4 \leq D \leq 8$) the luminosity in the lowest order is reproduced by string theory calculation as long as $c_{\text{eff}} = 6$.

4 Higher Order Contribution

A natural question at this point is what happens if we take account of not only the lowest order term in ω but also the higher order terms? For special cases, i.e., $D = 5$,

$n = 3$ [12] and $D = 4$, $n = 4$ [21], it was shown that the absorption cross section factorizes into Bose distribution functions,

$$\sigma_{\text{abs}} = |K|^2 |A|^2 = \frac{A_H T_H}{4 T_L T_R} \omega \frac{\rho(\omega/2T_L) \rho(\omega/2T_R)}{\rho(\omega/T_H)}, \quad D = 4, 5. \quad (4.1)$$

Note that the factor $|K|^2$ converts the absorption probability $|A|^2$ into the absorption cross section σ_{abs} . (The former is calculated for a spherical wave and the latter is for a plane wave.) So the emission rate is

$$\Gamma_{\text{sc}} = \sigma_{\text{abs}} \rho\left(\frac{\omega}{T_H}\right) \frac{d^{D-1}k}{(2\pi)^{D-1}} = \frac{A_H T_H}{4 T_L T_R} \omega \rho\left(\frac{\omega}{2T_L}\right) \rho\left(\frac{\omega}{2T_R}\right) \frac{d^{D-1}k}{(2\pi)^{D-1}}, \quad D = 4, 5. \quad (4.2)$$

Eqs.(4.2) and (3.4) agree including the constant factor, and so do eqs.(4.1) and (3.10), because $L\kappa_D/4 = A_H T_H/4 T_L T_R$ by eq.(2.15) and $A_H = 16 G_D S/f$.

This is the motivation for us to try to generalize [12] to arbitrary D and n . In the semiclassical calculation in D -dimensions, we will calculate the absorption cross section, which appears in eq.(4.2). In order to see an agreement with (3.10), σ_{abs} must take the following form

$$\sigma_{\text{abs}} = \frac{A_H T_H}{4 T_R T_L} \omega \frac{\rho(\omega/2T_L) \rho(\omega/2T_R)}{\rho(\omega/T_H)} \quad (4.3)$$

$$= A_H \left[1 + \frac{\omega^2}{48 T_L T_R} + O(\omega^3) \right], \quad \omega \rightarrow 0. \quad (4.4)$$

The complete factorization as in (4.3) might be difficult to see because one has to include all the ω dependence. Instead of eq.(4.3), we study the factorizability of σ_{abs} up to $O(\omega^2)$ for small ω as in eq.(4.4).

We study the propagation of a (neutral) scalar field ϕ in the background of a D -dimensional black hole carrying n charges,

$$ds_D^2 = -f^{-\frac{D-3}{D-2}} h dt^2 + f^{\frac{1}{D-2}} (h^{-1} dr^2 + r^2 d\Omega_{D-2}^2), \quad (4.5)$$

where

$$f = \prod_{i=1}^n \left(1 + \frac{r_i^{D-3}}{r^{D-3}} \right), \quad h = 1 - \frac{r_0^{D-3}}{r^{D-3}}. \quad (4.6)$$

r_i are charges. r_0 is the location of the horizon and it also works as a non-extremality parameter. ($r_0 = 0$ corresponds to the extremal limit.) This form of metric was first

introduced by Cvetič and Tseytlin [25]. (See also [26].) It covers various black hole configurations. For example, $n = 0$ gives the Schwarzschild black hole in D -dimensions. $D = 5, n = 3$ and $D = 4, n = 4$ black holes are, of course, special cases of eq.(4.5) [27]. As in the case of $D = 5, n = 3$ and $D = 4, n = 4$ black holes, this D -dimensional black hole is assumed to be obtained from a $(D + 1)$ -dimensional black string by compactifying the string direction on a circle. It is sometimes convenient to use another set of parameters, α_i and σ , instead of r_1, \dots, r_n ,

$$r_i^{D-3} = r_0^{D-3} \sinh^2 \alpha_i \quad (i = 1, \dots, n-1), \quad (4.7)$$

$$r_n^{D-3} = r_0^{D-3} \sinh^2 \sigma. \quad (4.8)$$

In the black string picture, r_n corresponds to the momentum flowing along the black string, which is obtained by applying a boost to the string. The boost is parameterized by σ . In the extremal limit $r_0 \rightarrow 0$, r_n is kept fixed, i.e., $\sigma \rightarrow \infty$. We want to solve the wave equation for ϕ in the dilute gas approximation [12],

$$r_0, r_n \ll r_1, \dots, r_{n-1}. \quad (4.9)$$

Under this approximation, the metric (4.5) describes a black hole with mass

$$M = M_0 + (D-3) \frac{\Omega_{D-2} r_0^{D-3}}{2\kappa_D^2} \cosh 2\sigma. \quad (4.10)$$

The first term $M_0 = [(D-1)r_0^{D-3}/2 + (D-3)\{r_1^{D-3} + \dots + r_{n-1}^{D-3}\}]\Omega_{D-2}/2\kappa_D^2$ corresponds to the ground state energy E_0 in eq.(2.9). The second term corresponds to the oscillator excitation, $E_L + E_R$ in eq.(2.9). Using the $(D-2)$ -dimensional area of the black hole horizon,

$$A_H = A_{D-2} = \frac{D-3}{4\pi} \frac{r_0^{D-3}}{T_H} \Omega_{D-2}, \quad (4.11)$$

one can calculate the Bekenstein–Hawking entropy,

$$S = (D-3) \frac{\Omega_{D-2} r_0^{D-3}}{2\kappa_D^2} \frac{1}{T_H}, \quad (4.12)$$

where the Hawking temperature T_H is given by

$$T_H = \frac{D-3}{4\pi} r_0^{-(D-3)\lambda} \left(\frac{r_0}{r_1 \cdots r_{n-1}} \right)^{(D-3)/2} \frac{1}{\cosh \sigma}. \quad (4.13)$$

From the black string point of view, the right moving oscillator excitations and the left moving ones are separately in equilibrium at temperatures;

$$T_R = \frac{D-3}{4\pi} r_0^{-(D-3)\lambda} \left(\frac{r_0}{r_1 \cdots r_{n-1}} \right)^{(D-3)/2} e^\sigma, \quad (4.14)$$

$$T_L = \frac{D-3}{4\pi} r_0^{-(D-3)\lambda} \left(\frac{r_0}{r_1 \cdots r_{n-1}} \right)^{(D-3)/2} e^{-\sigma}, \quad (4.15)$$

respectively. They give the connection between string theory quantities on the l.h.s. and black hole quantities on the r.h.s. The relation, eq.(2.14), is satisfied. Note that one can determine the effective length of the string L from eq.(2.15),

$$L = \frac{4}{\kappa_D^2} \frac{4}{f} \frac{\pi}{D-3} \left(r_0^{2\lambda} r_1 \cdots r_{n-1} \right)^{D-3} \Omega_{D-2}. \quad (4.16)$$

We look for a spherically symmetric configuration of a scalar field $\phi(t, r) = e^{-i\omega t} R(r)$, which satisfies the wave equation, $\sqrt{-g}^{-1} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \phi = 0$. $g_{\mu\nu}$ is given in eq.(4.5). The radial equation becomes,

$$\left[\frac{h}{r^{D-2}} \frac{d}{dr} h r^{D-2} \frac{d}{dr} + \omega^2 f \right] R(r) = 0. \quad (4.17)$$

It seems extremely difficult, however, to solve it for arbitrary r . Following [12], we solve it separately in two regions, i.e., in the near zone, $r \leq r_m$, and in the far zone, $r \geq r_m$, where the matching point r_m is chosen to satisfy

$$r_0, r_n \ll r_m \ll r_1, \dots, r_{n-1}. \quad (4.18)$$

Also we impose the low energy condition [12],

$$\omega \ll \frac{1}{r_1}, \dots, \frac{1}{r_{n-1}}. \quad (4.19)$$

In the far zone, eq.(4.17) becomes,

$$\frac{d^2 \psi}{d\rho^2} + \left[1 - \frac{(D-2)(D-4)}{4\rho^2} \right] \psi = 0, \quad (4.20)$$

where $\rho = \omega r$ and $R = r^{-(D-2)/2} \psi$. It has two independent solutions

$$F = \sqrt{\frac{\pi}{2}} \rho^{1/2} J_{(D-3)/2}(\rho), \quad (4.21)$$

$$G = \sqrt{\frac{\pi}{2}} \rho^{1/2} N_{(D-3)/2}(\rho), \quad (4.22)$$

which are the Bessel and Neumann functions respectively. The most general form of the solution is a linear combination of F and G with arbitrary constants α and β ,

$$R = r^{-(D-2)/2} (\alpha F + \beta G) = \sqrt{\frac{\pi}{2}} \omega^{1/2} r^{-(D-3)/2} [\alpha J_{(D-3)/2}(\omega r) + \beta N_{(D-3)/2}(\omega r)]. \quad (4.23)$$

In the limit $r \rightarrow \infty$, R becomes

$$R \simeq 2^{-1} r^{-(D-2)/2} [e^{-i\omega r} (\alpha e^{i(D-2)\pi/4} + \beta e^{iD\pi/4}) + e^{i\omega r} (\alpha e^{-i(D-2)\pi/4} + \beta e^{-iD\pi/4})]. \quad (4.24)$$

The first term in the square brackets is the incoming part while the second term is the outgoing part. Near the matching point r_m ,

$$R \simeq \sqrt{\frac{\pi}{2}} \left[\frac{\alpha \omega^{\frac{D-2}{2}}}{2^{\frac{D-3}{2}} \Gamma\left(\frac{D-1}{2}\right)} + \frac{\beta \omega^{1/2}}{\pi r^{\frac{D-3}{2}}} \left\{ \frac{\omega^{\frac{D-3}{2}}}{2^{\frac{D-5}{2}}} \frac{\gamma + \ln(\omega r/2)}{\Gamma\left(\frac{D-1}{2}\right)} - \frac{2^{\frac{D-3}{2}} \Gamma\left(\frac{D-3}{2}\right)}{(\omega r)^{\frac{D-3}{2}}} \right\} \right], \quad (4.25)$$

for D odd, and

$$R \simeq \sqrt{\frac{\pi}{2}} \left[\frac{\alpha \omega^{\frac{D-2}{2}}}{2^{\frac{D-3}{2}} \Gamma\left(\frac{D-1}{2}\right)} + \beta \frac{(-1)^{\frac{D+2}{2}} \omega^{\frac{1}{2}}}{\Gamma\left(\frac{D-1}{2}\right) r^{\frac{D-3}{2}}} \left(\frac{2}{\omega r}\right)^{\frac{D-3}{2}} \right], \quad (4.26)$$

for D even. The term multiplying α is common. Near the matching point, the term multiplying β becomes divergent for both D odd and even, under the low energy condition, eq.(4.19).

In the near zone,

$$\begin{aligned} f &\simeq \left[\prod_{i=1}^{n-1} \left(\frac{r_i}{r}\right)^{D-3} \right] \left(1 + \frac{r_n^{D-3}}{r^{D-3}}\right) \\ &= v^{n-1} \left(\frac{r_1 \cdots r_{n-1}}{r_0^{n-1}}\right)^{D-3} + v^n \left(\frac{r_1 \cdots r_n}{r_0^n}\right)^{D-3}. \end{aligned} \quad (4.27)$$

$v = (r_0/r)^{D-3}$ is a new radial variable. Eq.(4.17) becomes

$$(1-v) \frac{d}{dv} (1-v) \frac{dR}{dv} + v^{-2\lambda} \left(B + \frac{C}{v}\right) R = 0, \quad (4.28)$$

where

$$B = C \left(\frac{r_n}{r_0}\right)^{D-3} = C \sinh^2 \sigma, \quad (4.29)$$

$$C = \left(\frac{\omega r_0}{D-3}\right)^2 \left(\frac{r_1 \cdots r_{n-1}}{r_0^{n-1}}\right)^{D-3}. \quad (4.30)$$

B and C are related to the temperatures, eqs.(4.13), (4.14) and (4.15) by

$$\sqrt{B+C} = \frac{1}{4\pi} \frac{\omega}{T_H}, \quad C = \left(\frac{\omega}{4\pi}\right)^2 \frac{1}{T_L T_R}. \quad (4.31)$$

Near the horizon, $v \simeq 1$, we have the ingoing solution

$$R_{\text{in}} = e^{-i\sqrt{B+C}\log(1-v)} = (1-v)^{-i\sqrt{B+C}}, \quad (4.32)$$

which is independent of λ .

When r is in the near zone but not very close to the horizon r_0 , we assume the form of the solution to be,

$$R = R_0 z^A F(z), \quad z = 1 - v. \quad (4.33)$$

We require that $F \rightarrow 1$ as $z \rightarrow 0$. Thus $A = -i\sqrt{B+C}$. (See eq.(4.32).) R_0 is a normalization constant to be determined by the matching condition. We look for a solution in series expansion

$$F(z) = \sum_{n=0}^{\infty} b_n z^n, \quad b_0 = 1. \quad (4.34)$$

Substituting eqs.(4.33) and (4.34) into eq.(4.28), we have a recurrence relation for b_n ;

$$n(n+2A)b_n = - \sum_{m=0}^{n-1} \frac{1}{(n-m)!} [(k-1)_{n-m}B + (k)_{n-m}C] b_m \quad (n \geq 1), \quad (4.35)$$

where $k = 2\lambda + 1$. $(k)_n$ is the Pochhammer symbol; $(k)_n = k(k+1)\cdots(k+n-1)$, $(k)_0 = 1$. We expand b_n in powers of ω . $b_n^{(m)}$ denotes a term of order $O(\omega^m)$. Note $A \sim O(\omega)$ and $B \sim C \sim O(\omega^2)$. Up to second order in ω , eq.(4.35) becomes

$$b_n^{(0)} + b_n^{(1)} + b_n^{(2)} = - \sum_{m=0}^{n-1} \frac{1}{n^2(n-m)!} [(k-1)_{n-m}B + (k)_{n-m}C] b_m^{(0)} \quad (n \geq 1). \quad (4.36)$$

By comparing both sides of the equation order by order, we get $b_n^{(0)} = 0, b_n^{(1)} = 0, b_n^{(2)} = -B(k-1)_n/n^2n! - C(k)_n/n^2n! \quad (n \geq 1)$, or

$$b_0 = 1, \quad (4.37)$$

$$b_n = -\frac{(k-1)_n}{n^2n!}B - \frac{(k)_n}{n^2n!}C + O(\omega^3) \quad (n \geq 1). \quad (4.38)$$

R behaves near the matching point, $z \simeq 1$ (i.e. $v \simeq 0$), as

$$R = R_0 \sum_{n=0}^{\infty} b_n + O(v), \quad v \rightarrow 0. \quad (4.39)$$

The sum of b_n takes the following form;

$$\sum_{n=0}^{\infty} b_n = 1 - B(k-1) \sum_{n=1}^{\infty} \frac{(k)_{n-1}}{n^2 n!} - C \sum_{n=1}^{\infty} \frac{(k)_n}{n^2 n!} + O(\omega^3). \quad (4.40)$$

The matching condition is given by

$$R_0 \sum_{n=0}^{\infty} b_n = \sqrt{\pi} \alpha \left(\frac{\omega}{2} \right)^{\frac{D-2}{2}} \left[\Gamma \left(\frac{D-1}{2} \right) \right]^{-1}. \quad (4.41)$$

We dropped the β -dependent term by requiring $\alpha \gg \beta$. In order to calculate the absorption probability, we study the conserved flux, which is defined by

$$f = \frac{1}{2i} \left[R^* h r^{D-2} \frac{dR}{dr} - \text{c.c.} \right]. \quad (4.42)$$

The incoming flux from infinity is given by the first term of eq.(4.24),

$$f_{\text{in}} = -\frac{\omega}{4} |\alpha|^2. \quad (4.43)$$

The absorbed flux is calculated from eqs.(4.33) and (4.34) at the horizon, $z \simeq 1$. One gets

$$f_{\text{abs}} = -\frac{D-3}{4\pi} \frac{r_0^{D-3}}{T_H} \omega |R_0|^2. \quad (4.44)$$

We used eq.(4.31). Absorption probability is computed from eqs.(4.41), (4.43) and (4.44),

$$|A|^2 = \frac{f_{\text{abs}}}{f_{\text{in}}} = \frac{D-3}{4\pi} \frac{r_0^{D-3}}{T_H} \frac{\pi \omega^{D-2}}{2^{D-4}} \left[\Gamma \left(\frac{D-1}{2} \right) \right]^{-2} \left| \sum_{n=0}^{\infty} b_n \right|^{-2}. \quad (4.45)$$

Using the factor $|K|^2$, one can calculate the absorption cross section,

$$\sigma_{\text{abs}} = |K|^2 |A|^2 = A_H \left| \sum_{n=0}^{\infty} b_n \right|^{-2}. \quad (4.46)$$

A_H is given in eq.(4.11). Combining eq.(4.40) with eq.(4.46), one gets

$$\sigma_{\text{abs}} = A_H \left[1 + 2B(k-1) \sum_{n=1}^{\infty} \frac{(k)_{n-1}}{n^2 n!} + 2C \sum_{n=1}^{\infty} \frac{(k)_n}{n^2 n!} + O(\omega^3) \right]. \quad (4.47)$$

This should be compared with eq.(4.4),

$$\sigma_{\text{abs}} = A_H \left[1 + 2C\zeta(2) + O(\omega^3) \right]. \quad (4.48)$$

We used eq.(4.31) and $\zeta(2) = \pi^2/6$.

Notice that there is no B (or r_n) dependence in eq.(4.48). Therefore, the B term in eq.(4.47) must vanish;

$$(k-1) \sum_{n=1}^{\infty} \frac{(k)_{n-1}}{n^2 n!} = 0. \quad (4.49)$$

This equation has at least one solution, $k = 1$.

0) When $k = 1$, eq.(4.47) becomes $\sigma_{\text{abs}} = A_H[1 + 2C\zeta(2) + O(\omega^3)]$, i.e., eq.(4.48) is satisfied.

This corresponds to $\lambda = 0$, i.e., $D = 5$, $n = 3$ and $D = 4$, $n = 4$ cases [12][21].

If there is another value of k_0 which realizes the factorization, it has to satisfy the following two equations,

$$\sum_{n=1}^{\infty} \frac{(k_0)_{n-1}}{n^2 n!} = 0, \quad (4.50)$$

$$\sum_{n=1}^{\infty} \frac{(k_0)_n}{n^2 n!} = \zeta(2). \quad (4.51)$$

The first condition eliminates the B -dependent term in eq.(4.47). The second condition gives the right coefficient to the $O(\omega^2)$ term.

1) When $k_0 \geq 0$ ($\lambda \geq -1/2$), eq.(4.50) is not satisfied because the l.h.s. becomes a sum of positive numbers.

2) When $-1 < k_0 < 0$ ($-1 < \lambda < -1/2$), eq.(4.51) is not satisfied because $k_0 < 0$ and $k_0 + 1 > 0$ mean,

$$\sum_{n=1}^{\infty} \frac{(k_0)_n}{n^2 n!} = k_0 \sum_{n=1}^{\infty} \frac{(k_0 + 1)_{n-1}}{n^2 n!} < 0. \quad (4.52)$$

3) When k_0 is a negative integer ($\lambda = -1, -3/2, -2, -5/2, \dots$), the sum in eq.(4.51) becomes a finite sum of rational numbers, which cannot be equal to an irrational number $\zeta(2)$.

1) and 3) eliminate all the possibilities for $D = 4, 5$ except for $\lambda = 0$ (from result 0)).

In fact,

$$\lambda_{D=4} = 2 - \frac{n}{2} = 2, \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, \dots \quad (4.53)$$

$$\lambda_{D=5} = \frac{3}{2} - \frac{n}{2} = \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, \dots \quad (4.54)$$

As far as we know, $k = 1$ ($\lambda = 0$) is the only possibility when factorization occurs. For negative non-integral k_0 's, possibilities of factorization have not been eliminated. However, it seems unlikely that eqs.(4.50) and (4.51) are both satisfied.

5 Summary

We have compared the string calculation at weak coupling with the semiclassical calculation at strong coupling. The following is a summary of what we have learned.

- a) From the lowest order in ω calculation, we found that the Bekenstein–Hawking entropy is reproduced for arbitrary D (and n) provided that $c_{\text{eff}} = 6$.
- b) When $\lambda = 0$ (i.e., $D = 5$, $n = 3$ or $D = 4$, $n = 4$), there is agreement between the string calculation and the semiclassical one to all orders in ω with $c_{\text{eff}} = 6$ [12][21].
- c) For $\lambda > 0$, $-1 < \lambda < 0$, $\lambda = -m/2$ ($m = 2, 3, 4, \dots$), the emission rate in the string calculation and that in the semiclassical calculation don't agree beyond the lowest order in ω .

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References

- [1] L. Susskind, hep-th/9309145.
- [2] A. Sen, Mod. Phys. Lett. **A10**, 2081 (1995), hep-th/9504147.
- [3] A. Strominger and C. Vafa, Phys. Lett. **B379**, 99 (1996), hep-th/9601029.
- [4] C. G. Callan, Jr. and J. M. Maldacena, Nucl. Phys. **B472**, 591 (1996), hep-th/9602043.

- [5] G. T. Horowitz and A. Strominger, Phys. Rev. Lett. **77**, 2368 (1996), hep-th/9602051.
- [6] J. Breckenridge, D. Lowe, R. Myers, A. Peet, A. Strominger and C. Vafa, Phys. Lett. **B381**, 423 (1996), hep-th/9603078.
- [7] J. Maldacena and A. Strominger, Phys. Rev. Lett. **77**, 428 (1996), hep-th/9603060.
- [8] C. Johnson, R. Khuri and R. Myers, Phys. Lett. **B378**, 78 (1996), hep-th/9603061.
- [9] G. Horowitz, D. Lowe and J. Maldacena, Phys. Rev. Lett. **77**, 430 (1996), hep-th/9603195.
- [10] J. Maldacena, hep-th/9611125.
- [11] S. R. Das and S. D. Mathur, hep-th/9606185.
- [12] J. Maldacena and A. Strominger, hep-th/9609026.
- [13] R. Kallosh, A. Linde, T. Ortin, A. Peet, and A. Van Proeyen, Phys. Rev. **D46**, 5278 (1992), hep-th/9205027.
- [14] M. Cvetič and D. Youm, Phys. Rev. **D53**, 584 (1996), hep-th/9507090.
- [15] E. Halyo, B. Kol, A. Rajaraman and L. Susskind, hep-th/9609075.
- [16] S. R. Das, G. Gibbons and S. D. Mathur, hep-th/9609052.
- [17] E. Halyo, A. Rajaraman and L. Susskind, hep-th/9605112.
- [18] J. M. Maldacena, Ph. D thesis, Princeton University 1996, hep-th/9607235.
- [19] A. A. Tseytlin, Nucl. Phys. **B477**, 431 (1996), hep-th/9605091.
- [20] F. Larsen and F. Wilczek, hep-th/9609084.
- [21] S. S. Gubser and I. R. Klebanov, hep-th/9609076.

- [22] S. R. Das and S. D. Mathur, Phys. Lett. **B375**, 103 (1996), hep-th/9601152.
- [23] J. Maldacena and L. Susskind, Nucl. Phys. **B475**, 679 (1996), hep-th/9604042.
- [24] S. R. Das and S. D. Mathur, hep-th/9607149.
- [25] M. Cvetič and A. A. Tseytlin, Nucl. Phys. **B478**, 181 (1996), hep-th/9606033.
- [26] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. **B475**, 164 (1996), hep-th/9604089.
- [27] G. T. Horowitz, J. M. Maldacena and A. Strominger, Phys. Lett. **B383**, 151 (1996), hep-th/9603109.